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Quantum kink model and SU(2) symmetry: spin interpretation and T-violation

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Abstract. In this paper we consider the class of exact solutions of the Schrödinger equation with the Razavi potential. By means of this we obtain some wavefunctions and mass spectra of the relativistic scalar field model with spontaneously broken symmetry near the static kink solution. Appearance of the bosons, which have two different spins, will be shown in the theory, thereby the additional breaking of discrete symmetry between the quantum mechanical kink particles with the opposite spins (i.e. the T-violation) takes place.

At present, quantum field theories, having topologically non-trivial solutions, are being intensively developed. The mass spectra of particles, which are predicted by such theories, can be received by means of the effective action formalism [1], which describes the low-energy dynamics of stable solutions taking into account small quantum oscillations. In particular, [2] was devoted to one such theory, namely, the $d = 1 + 1$ relativistic model φ^4 with spontaneously broken symmetry. In that paper the non-perturbative quantum scalar field theory near the static kink solution, which can be interpreted as a quantum mechanical heavy particle, was considered. As a result of quantization of the kink's internal degrees of freedom, the Schrödinger equation was received in terms of the raising and lowering operators. It was noted that, dependent on what ordering procedure for the operators was chosen, unitary non-equivalent theories take place. Regrettably, important aspects of the physical sense of such theories, as well as the question of obtaining exact solutions and mass spectra, remain open. In this paper we try to resolve these problems in particular. It became possible owing to the analogies found between the key equations of [2] and the wide class of the Schrödinger equations with the double-well potentials related to SU(2) symmetry [4–6], in particular, the Razavi potentials [7–9].

We start from the action

$$S[\varphi] = \int \mathrm{d}^2x \left\{ \frac{1}{2}\frac{\partial\varphi}{\partial x^i}\frac{\partial\varphi}{\partial x_i} - \frac{1}{4}g^2\left[\varphi^2 - \left(\frac{m}{g}\right)^2\right]^2 \right\} \tag{1}$$

where $\varphi(x,t)$ is the dimensionless scalar field, m and g are real parameters. The corresponding equations of motion have the kink solution [3]

$$\varphi_c(x) = \frac{m}{g}\tanh\frac{mx}{\sqrt{2}} \tag{2}$$

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with the non-trivial behaviour at infinity

$$\varphi_c(+\infty) = -\varphi_c(-\infty) = \frac{m}{g} \tag{3}$$

and the non-zero topological charge

$$Q = \frac{g}{m}\int_{-\infty}^{+\infty} \frac{\partial\varphi(x)}{\partial x}\,\mathrm{d}x = \frac{g}{m}[\varphi(x=+\infty) - \varphi(x=-\infty)]. \tag{4}$$

We perform transformation on the new set of the variables

$$\begin{aligned} &x^m = x^m(s) + e^m_{(1)}(s)\rho \\ &\varphi(x,t) = \tilde{\varphi}(\sigma_a) \qquad \sigma_{a=0} = s \qquad \sigma_{a=1} = \rho \end{aligned} \tag{5}$$

where s and ρ are the so-called collective coordinates, $x^m(s)$ turn out to be the coordinates of a $(1+1)$-dimensional point particle, $e^m_{(1)}(s)$ is the unit spacelike vector, which is orthogonal to a worldline of the particle. It should be pointed out that unlike (x,t) the new base variables (s,ρ) are invariant under the Poincaré transformations.

Considering field $\tilde{\varphi}(\sigma_a)$ excitations near the kink (2) and eliminating zero modes, it is possible to obtain the non-minimal p-brane (more strictly, non-minimal $(1+1)$-dimensional point particle with curvature) action as a residual effective action for the model (1), see [2] for details,

$$S_{\text{eff}} = -\mu\int \mathrm{d}s\,\sqrt{\dot{x}^2}\left(1 + \frac{k^2}{3m^2}\right) \tag{6}$$

$$\mu = \frac{2\sqrt{2}}{3}\frac{m^3}{g^2} \tag{7}$$

where $k = \sqrt{-a^i a_i}$ is the curvature of a point-particle worldline, a_n is the acceleration

$$a_n = \frac{1}{\sqrt{\dot{x}^2}}\frac{\mathrm{d}}{\mathrm{d}s}\frac{\dot{x}_n}{\sqrt{\dot{x}^2}}.$$

From (6) it follows that we have obtained a theory with higher derivatives. In this theory we have two pairs of canonical variables $\{x_m, p_m\}$ and $\{q_m = \dot{x}_m, \Pi_m\}$ which are constrained on a certain submanifold of the total phase space by both the two primary first-kind constraints $\Phi_{1,2} \approx 0$ and the proper time-gauge condition $\sqrt{q_m q^m} \approx 1$. After some transformations one of the constraints can be rewritten as

$$\Phi_2 = -\sqrt{p^2}\cosh v + \mu - \frac{\mu}{\xi^2}\Pi_v^2 \approx 0 \tag{8}$$

$$\xi = \frac{2}{\sqrt{3}}\frac{\mu}{m} \tag{9}$$

where v is the new coordinate, Π_v is the corresponding momentum, which are interpreted in paper [2] as the spin values. Below it will be shown that this interpretation is not complete and the true SU(2) spin operators will be introduced.

In the quantum case the condition $\hat{\Phi}_2|\Psi\rangle = 0$, where $\hat{\Pi}_v = -\mathrm{i}\partial/\partial v$ is the momentum operator in the coordinate representation and $\Psi = \Psi(v)$ is the wavefunction of the kink, must be satisfied. The constraint $\hat{\Phi}_2$ permits two modifications, consideration of which gives us the equations of motion in terms of the raising and lowering operators

$$[a^\dagger_\lambda a_\lambda - \xi^2(1-\lambda)]\Psi(v) = 0 \tag{10}$$

$$[a_\lambda a^\dagger_\lambda - \xi^2(1-\lambda)]\Psi(v) = 0 \tag{11}$$

where

$$a_\lambda = \frac{\mathrm{d}}{\mathrm{d}v} + \sqrt{2\lambda}\xi \sinh\frac{v}{2} \tag{12}$$

$$\lambda = \frac{\sqrt{p^2}}{\mu} = \frac{M}{\mu}. \tag{13}$$

These equations can be written in the form of a Schrödinger equation

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}v^2} + 2\lambda\xi^2\sinh^2\frac{v}{2} - \sqrt{\frac{\lambda}{2}}\xi\cosh\frac{v}{2}\right)\Psi(v) = \xi^2(1-\lambda)\Psi(v) \tag{14}$$

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}v^2} + 2\lambda\xi^2\sinh^2\frac{v}{2} + \sqrt{\frac{\lambda}{2}}\xi\cosh\frac{v}{2}\right)\Psi(v) = \xi^2(1-\lambda)\Psi(v). \tag{15}$$

In [2] only equation (14) was considered, for which the wavefunction of the ground (vacuum) state was found

$$\begin{aligned} &a_{\lambda=1}\Psi_{\mathrm{vac}}(v) = 0 \\ &\Psi_{\mathrm{vac}}(v) = C\exp\left(-2\sqrt{2}\xi\cosh\frac{v}{2}\right). \end{aligned} \tag{16}$$

Below we represent the approach, which helps us to take a new look at the expressions (10)–(15) as well as to obtain some exact results and deeper interpretation of the theory.

Let us consider the Schrödinger equation [7–9]

$$[\hat{H} - \varepsilon]\Psi(\zeta) = 0 \tag{17}$$

where

$$\hat{H} = -\frac{\mathrm{d}^2}{\mathrm{d}\zeta^2} + \frac{B^2}{4}\sinh^2\zeta - B\left(S + \frac{1}{2}\right)\cosh\zeta. \tag{18}$$

Here S and B are dimensionless parameters. It can readily be shown that SU(2) is the dynamic group of symmetry for this Hamiltonian and to provide the direct analogy with the spin Hamiltonian

$$\hat{H}_s = -S_z^2 - BS_x \tag{19}$$

using the information induced by the su(2) Lie algebra [10].

On a subset $L^2(R)$ the following spin operators act

$$S_x = S\cosh\zeta - \frac{B}{2}\sinh^2\zeta - \sinh\zeta\frac{\mathrm{d}}{\mathrm{d}\zeta} \tag{20}$$

$$S_y = \mathrm{i}\left\{-S\sinh\zeta + \frac{B}{2}\sinh\zeta\cosh\zeta + \cosh\zeta\frac{\mathrm{d}}{\mathrm{d}\zeta}\right\} \tag{21}$$

$$S_z = \frac{B}{2}\sinh\zeta + \frac{\mathrm{d}}{\mathrm{d}\zeta}. \tag{22}$$

Thereby the commutation relations

$$[S_i, S_j] = \mathrm{i}\epsilon_{ijk}S_k \tag{23}$$

$$S_x^2 + S_y^2 + S_z^2 \equiv S(S+1) \tag{24}$$

are valid.

Now we consider the case $S \geqslant 0$. Then an irreducible finite-dimensional subspace of representation space of the SU(2) algebra, which is invariant with respect to the operators (20)–(22), exists. Its dimension is $2S + 1$.

One can verify [8] that the solution of (17) is the function

$$\Psi(\zeta) = \exp\left(-\frac{B}{2}\cosh\zeta\right)\sum_{\sigma=-S}^{S}\frac{c_\sigma}{\sqrt{(S-\sigma)!(S+\sigma)!}}\exp(\sigma\zeta) \tag{25}$$

where the coefficients c_σ satisfy with the system of linear equations

$$\begin{aligned}&(\varepsilon+\sigma^2)c_\sigma + \frac{B}{2}\left[\sqrt{(S-\sigma)(S+\sigma+1)}\; c_{\sigma+1} + \sqrt{(S+\sigma)(S-\sigma+1)}\; c_{\sigma-1}\right] = 0\\ &c_{S+1} = c_{-S-1} = 0 \qquad \sigma = -S, -S+1, \ldots, S.\end{aligned} \tag{26}$$

The solution of this system is equivalent to the determination of eigenvectors and eigenvalues of the operator $\hat{H}$ in the matrix representation, which is realized in a finite-dimensional subspace of the su(2) Lie algebra. Analytical solutions of equations (25) and (26) were found for the following spin values.

(i) $S = 0$. The dimension of the invariant subspace of the algebra is equal to 1, therefore, only one wavefunction and ground-state energy can be found. We have

$$\Psi_0(\zeta) = A_0\exp\left(-\frac{B}{2}\cosh\zeta\right) \qquad \varepsilon_0 = 0. \tag{27}$$

(ii) $S = \frac{1}{2}$. We obtain two wavefunctions and energies of according states

$$\begin{aligned}\Psi_0(\zeta) &= A_0\exp\left(-\frac{B}{2}\cosh\zeta\right)\cosh\left(\frac{1}{2}\zeta\right) \qquad \varepsilon_0 = -\frac{B}{2} - \frac{1}{4}\\ \Psi_1(\zeta) &= A_1\exp\left(-\frac{B}{2}\cosh\zeta\right)\sinh\left(\frac{1}{2}\zeta\right) \qquad \varepsilon_1 = \frac{B}{2} - \frac{1}{4}.\end{aligned} \tag{28}$$

(iii) $S = 1$. There are solutions for three lower levels

$$\begin{aligned}\Psi_0(\zeta) &= A_0\exp\left(-\frac{B}{2}\cosh\zeta\right)\left(1 - \frac{\varepsilon_0}{B}\cosh\zeta\right) \qquad \varepsilon_0 = -\frac{r_+}{2}\\ \Psi_1(\zeta) &= A_1\exp\left(-\frac{B}{2}\cosh\zeta\right)\sinh\zeta \qquad \varepsilon_1 = -1\\ \Psi_2(\zeta) &= A_2\exp\left(-\frac{B}{2}\cosh\zeta\right)\left(1 - \frac{\varepsilon_2}{B}\cosh\zeta\right) \qquad \varepsilon_2 = -\frac{r_-}{2}\end{aligned} \tag{29}$$

where $r_\pm = 1 \pm \sqrt{1+4B^2}$, A_i are integration constants.

(iv) $S = \frac{3}{2}, 2$. In this work these cases are not considered, however, it is still possible to find exact solutions for them [8].

(v) $S > 2$, $2S$ is an integer. Since it is impossible to solve the system (26) exactly, there are not any analytical solutions in this case.

(vi) Either $2S$ is a non-integer or $S < 0$. For such spin values an invariant subspace of the algebra does not exist [9].

Now we apply the results obtained above to the $d = 1 + 1$ relativistic model φ^4 with spontaneously broken symmetry near the static kink solution. Indeed, it is easy to show that equations (14) and (15) can be rewritten in the form (17), if we suppose

$$v = 2\zeta \qquad B = 4\sqrt{2\lambda}\xi \qquad \varepsilon = 4\xi^2(1-\lambda)$$

and $S = 0$ for the equation (14), $S = -1$ for (15).

For $S = 0$ from equations (9), (13) and (27) we obtain the mass spectrum of the kink boson in the ground state. It equals the spectrum for the case of a free particle with the mass μ

$$M^{(S=0)}_{(n=0)} = \mu \tag{30}$$

and the known result (16) can be obtained.

It is easy to see that the symmetry between the particles with the spins $S = 1$ and $S = -1$ is broken in the studied theory. Indeed, for $S = 1$ equations (9), (13) and (29) yield

$$
\begin{aligned}
M^{(S=1)}_{(n=0)} &= \mu - \frac{21}{32}\frac{m^2}{\mu} \pm \frac{27m}{32}\sqrt{\left(\frac{m}{\mu}\right)^2 + \frac{512}{243}} \\
M^{(S=1)}_{(n=1)} &= \mu + \frac{3}{16}\frac{m^2}{\mu} \\
M^{(S=1)}_{(n=2)} &= \mu + \frac{27}{32}\frac{m^2}{\mu} \pm \frac{27m}{32}\sqrt{\left(\frac{m}{\mu}\right)^2 + \frac{512}{243}}
\end{aligned}
\tag{31}
$$

where the signs '$\pm$' denote additional splitting of even mass spectra at least for lower states. Therefore, in this theory the breaking of discrete symmetry under the time inversion (the T-violance) takes place [11] additionally to the symmetry breaking of the initial theory (1).

It should also be noted that the true spin operator of the model is not the operator of the canonical momentum $\widehat{\Pi}_v$. The true spin operators are given by the expressions (20)–(22), thereby for the case $S = 0$ they are equal to S_z up to the factors $-\sinh\zeta$ and $\mathrm{i}\cosh\zeta$ respectively.

Finally, we represent another important aspect. It is now evident that the main demand on models such as that of [2] is their correspondence to reality. Otherwise all these particle-like solutions and field theories based on them will be no more than interesting mathematical toys, i.e. 'physics for one day'. As for the model considered here the analogy with the spin Hamiltonians is very useful in this connection. It is well known that both the Hamiltonians like (19) and double-well potentials are often exploited in physics. As examples one can point out the following applications, the anisotropic paramagnet [12], the theory of molecular vibrations [13], the model of the anharmonic oscillator in field theories [14], and the model of interacting fermions in nuclear physics [15].

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